A Transformation Approach to the Terminal Control Problem

G. L. Slater*
University of Cincinnati, Cincinnati, Ohio

The application of optimal control theory to the terminal control problem is investigated. It is shown that a transformation to an alternate problem yields additional insight into the structure of the optimal control, and presents a convenient computational tool for analytical and numerical studies. In particular, if the performance index is a function of only a few of the state vector components, a significant reduction in the numerical effort is possible, and the Riccati matrix can be obtained by a simple numerical quadrature. Application to a missile guidance problem is analyzed.

I. Introduction

In this paper we consider the linear-quadratic control problem: find u to minimize†

$$J=\frac{1}{2}(x_f-z_f)'M(x_f-z_f)+\frac{1}{2}\int_0^{t_f}(u'Nu)dt \tag{1}$$

Subject to the constraint

$$\dot{x}=Ax+Bu, \quad x(0)=x_0 \tag{2}$$

where $x \epsilon R_n$, $u \epsilon R_m$, M is an $n\times n$ non-negative matrix of rank k and N is an $m\times m$ positive definite matrix of full rank. In Eq. (1), $x_f\equiv x(t_f)$, and $z_f \epsilon R_n$ is the "desired" terminal state vector.

Such a problem, where only the final value of the state is included in the performance index, is referred to as "the terminal control problem." This problem is applicable, e.g., to guidance and control problems where only the final position and/or velocity vector is of interest. Although the solution of the foregoing problem can be obtained easily as a special case of the "linear-quadratic problem" (see, e.g., Ref. 1), some of the interesting aspects of the terminal control problem have made it a topic of special study.[2,3]

The intent of this paper is to explore some of these additional features which often will allow analytical solutions, and may serve to reduce computational effort. In this paper it will be assumed that terminal constraints of the form $\psi x_f=0$ will be handled by letting appropriate elements of M be arbitrarily large, even though the methods discussed here can be easily extended to handle the constraints exactly. In realistic control problems such exact constraints are rarely used since they generally lead to infinite feedback gains as the time-to-go approaches zero.

II. Conventional Solution

The solution of the terminal control problem can be obtained directly from the maximum principle.[4] Define the Hamiltonian

$$H=\tfrac{1}{2}u'Nu+\lambda'(Ax+Bu) \tag{3}$$

Then the optimal control u^* to minimize H is

$$u^*=-N^{-1}B'\lambda \tag{4}$$

where

$$\dot{\lambda}=-\partial H/\partial x=-A'\lambda \tag{5}$$

with boundary condition

$$\lambda(t_f)=M(x(t_f)-z_f) \tag{6}$$

For the feedback controller, assume a solution for λ in the form

$$\lambda(t)=S(t)x(t)+R(t)z_f \tag{7}$$

Differentiating (6), and substituting the preceding relations yields

$$\dot{S}=-A'S-SA+SBN^{-1}B'S$$
$$S(T_f)=M \tag{8}$$
$$\dot{R}=-A'R+SBN^{-1}B'R$$
$$R(t_f)=M \tag{9}$$

Thus S is the solution of the Riccati Eq. (8), while R is obtained from the linear (time-varying) Eq. (9). Equations (8) and (9) can be solved numerically in straightforward fashion although analytic solutions to even the simplest cases can be difficult.

Rather than solve for $R(t)$, let $R(t)z_f=-S(t)z(t)$, where $z(t)$ is an unknown vector. Differentiating and substituting Eqs. (8) and (9) yield the simple result

$$\dot{z}=Az$$
$$z(t_f)=z_f \tag{10}$$

The optimal feedback controller then is

$$u^*=-N^{-1}B'S(t)[x-z]. \tag{11}$$

The interpretation of this $z(t)$ is that it is the required value of $x(t)$, such that with zero control, the system free motion will bring the state to the desired value z_f. Thus the control at any time is proportional to the difference between the true state and the desired state at that time.

III. Transformation Approach

The previous result can be interpreted in an alternate way, and one which is often more amenable to computation, by

Received December 12, 1975; revision received May 14, 1976.
Index category: Navigation, Control, and Guidance Theory.

*Assistant Professor, Dept. of Aerospace Engineering. Member AIAA.

†( )′ is used to indicate the transpose of a vector or matrix quantity.

transforming to an alternate problem. Define the $n \times n$ square matrix $\Lambda(t)$ by

$$\dot{\Lambda}(t) = -\Lambda A$$

$$\Lambda(t_f) = I \tag{12}$$

Λ is termed the adjoint matrix. It is the transpose of the fundamental matrix of the system adjoint to Eq. (2).

Also let

$$\eta(t) = \Lambda(t)x(t) - z_f \tag{13}$$

Then obviously

$$\eta_f = \eta(t_f) = x_f - z_f \tag{14}$$

and by differentiating Eq. (13) we see that η satisfies the simple differential equation

$$\dot{\eta} = B_2(t)u \tag{15}$$

where

$$B_2(t) = \Lambda(t)B$$

From Eqs. (14) and (15) we see that $\eta(t)$ can be interpreted as the terminal miss if control is terminated at time t.

Using Eqs. (1) and (14) the performance index can be written

$$\frac{1}{2}\eta_f' M \eta_f + \frac{1}{2}\int_0^{t_f} u' N u dt$$

This optimization problem can be solved by defining a new Hamiltonian

$$H_2 = \frac{1}{2}u' N u + \lambda_2' B_2 u$$

which yields

$$u^* = -N^{-1} B_2' \lambda_2(t)$$

$$\lambda_2(t) = M\eta(t_f) \quad \text{(a constant)}$$

Hence

$$u^*(t) = -N^{-1} B_2' M \eta(t_f) \tag{16}$$

We can formally construct a feedback controller by defining

$$\lambda_2(t) = T(t)\eta(t) \tag{17}$$

where the matrix $T(t)$ can be shown to satisfy

$$\dot{T} = TKT, \quad T(t_f) = M \tag{18}$$

where

$$K = B_2(t) N^{-1} B_2'(t) = \Lambda B N^{-1} \beta' \Lambda' \tag{19}$$

Then of course

$$u^*(t) = -(N^{-1} B_2' T)\eta \tag{20}$$

which shows that the feedback control is always proportional to the expected terminal miss. Substituting for $\eta(t)$ from Eq. (13) yields

$$u^*(t) = -N^{-1} B_2' T(\Lambda x - z_f) = -N^{-1} B'(\Lambda' T \Lambda)[x - \Lambda^{-1} z_f]$$

Now $\Lambda^{-1} = \Phi$ is a fundamental matrix for Eq. (10), hence $\Lambda^{-1} z_f = z(t)$ of Eq. (10). The matrix $\Lambda' T \Lambda$ can be shown to satisfy Eq. (8), the equation for the Riccati matrix S, hence we see that the control in Eq. (20) is identical to that of the untransformed system.

In view of the simplicity of this sytem, the work required to obtain the Riccati equation solution is hardly justified. Substitute Eq. (16) into Eq. (15) to obtain

$$\dot{\eta} = -B_2 N^{-1} B_2' M \eta(t_f)$$

Hence, by quadrature,

$$\eta(t) = \eta(t_f) + \left[\int_t^{t_f} B_2 N^{-1} B_2' d\tau\right] M \eta(t_f)$$

$$= \left[I + \left(\int_t^{t_f} B_2 N^{-1} B_2' d\tau\right) M\right] \eta(t_f) \tag{21}$$

Then

$$M\eta(t_f) = M\left[I + \left(\int_t^{t_f} B_2 N^{-1} B_2' d\tau\right) M\right]^{-1} \eta(t) \tag{22a}$$

or if M^{-1} exists

$$= \left[M^{-1} + \int_t^{t_f} B_2 N^{-1} B_2' d\tau\right]^{-1} \eta(t) \tag{22b}$$

can be used to obtain the feedback solution. The matrix multiplying $\eta(t)$ in Eq. (22) is of course just the Riccati matrix $T(t)$.

If the original equation of motion had been

$$\dot{x} = Ax + Bu + Cv(t) \tag{23}$$

where $v(t)$ is a *known* forcing function, then the transformation Eq. (13) can be modified to

$$\eta(t) = \Lambda(t)x(t) + \int_t^{t_f} \Lambda(\tau) C v(\tau) d\tau - z_f \tag{24}$$

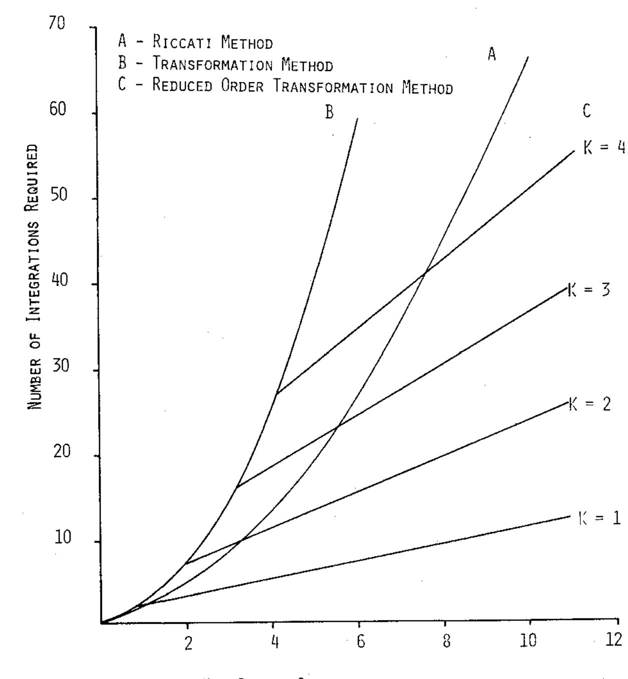

Fig. 1 Comparison of numerical effort involved to find optimal control solution.

By differentiating Eq. (24) it is readily seen that $\eta(t)$ satisfies Eqs. (14)-(15) as before, hence this problem is solved in similar fashion. Again the interpretation is that $\eta(t)$ is the deviation from the desired final state, z_f at the terminal time, t_f if the control u were terminated at time t.

In the real world, solution of this problem is complicated by the fact that future disturbances are rarely known for all time as implied by Eq. (24), although the "optimal" solution clearly demands it. There are generally three ways out of this dilemma, each of which is appealing in certain cases. These are: 1) Assume a predictable disturbance. Generally $v(t)=0$ is assumed. After all this is why people go to a feedback controller rather than the more easily obtained open loop controller. 2) Assume $v(t)$ is governed by some random statistical process. A feedback controller that gives the best "average" performance can be obtained if the statistics of the disturbance function are known. 3) Assume $v(t)$ is governed by an intelligent adversary who tries to maximize your index of performance. This yields a differential game format but one which is still deterministic in nature. Solutions of (2) and (3) are obviously not treated in this note.

IV. Reduction in Dimensionality

If rank $M=k<n$, then use of the transformation approach allows a further reduction in complexity and computational requirements. Consider the system as before where M is of the special form

$$M=\begin{bmatrix} d_1^2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & d_2^2 & & & & 0 \\ \cdot & & \cdot & & & \cdot \\ & & & d_k^2 & & \cdot \\ & & & & 0 & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \tag{25}$$

This is the case, for example, in an intercept problem when we desire to minimize the "miss distance" and do not care about the other components of the state vector. If M is of rank k, then an appropriate choice of state variables always will lead to this form of M.‡

Define the matrices

$$E=[I_k \,\vdots\, 0], \tag{26}$$

$$D=\begin{bmatrix} d_1^2 & & & 0 \\ & d_2^2 & & \\ & & \ddots & \\ 0 & & & d_k^2 \end{bmatrix} \tag{27}$$

then M can be written

$$M=E'DE$$

From Eq. (13) define a k-dimensional "reduced" state vector

$$\eta_R=E\eta=\Lambda_R x-z_{R_f} \tag{29}$$

where

$$\Lambda_R=E\Lambda(t)$$

$$z_{R_f}=Ez_f$$

the reduced quantities satisfy

$$\dot{\Lambda}_R=-\Lambda_R A \tag{30}$$

$$\Lambda_R(t_f)=E$$

$$\dot{\eta}_R=(\Lambda_R B)u=B_3(t)u \tag{31}$$

and the performance index becomes

$$J=\frac{1}{2}\eta_R' D\eta_R\Big|_{t_f}+\frac{1}{2}\int_0^{t_f}u'Nudt$$

Since η_R is now a k-vector, the optimal solution using Eq. (29) can be significantly easier than in the original transformation method. For example, the conventional Riccati solution will require $n(n+1)/2+n$ integrations to determine u^*. The transformation method requires n^2 integrations to determine Λ plus an addition $n(n+1)/2$ integrations for u^*. The reduced order transformation technique on the other hand uses nk integrations for Λ and an additional $k(k+1)/2$ integrations for the control. Thus, while the transformation method is extravagant in the number of integrations required (approximately a factor of three greater for large n), the reduced order method will be significantly better for small k as shown in Fig. 1. The reduced order transformation will result in fewer integrations when $k\leq n/2$ for small $n(\leq 10)$ and when $k\leq n/3$ for large n.

V. Applications

The primary advantage of the transformation approach is as a numerical scheme for solving terminal control problems of high order. Thus, for example, in large order systems, such as econometric and distribution models, the numerical effort required for solution may be significantly reduced at the expense of an increase in required storage. An alternate advantage which is perhaps of no less importance is that this technique permits calculation of analytic solutions to the problems of "moderate" complexity. While analytic solutions are not particularly important in an operational sense, they are useful in understanding the structure of the terminal controller and in building approximate control models.

As an application in this spirit consider a missile pursuit problem as indicated in Fig. 2. Using a missile centered rotating coordinate system and linearizing about the nominal pursuit triangle we obtain the kinematic equations[5]

$$\delta\dot{x}=-V_T\cos\beta_0\delta\beta+y_0(t)U_M$$

$$\delta\dot{y}=-V_T\sin\beta_0\delta\beta-x_0(t)U_m$$

$$\delta\dot{\beta}=-U_m$$

$$y_0(t)=(V_M-V_T\cos\beta_0)(t_f-t)$$

$$x_0(t)=V_T\sin\beta_0(t_f-t)$$

where U_m is the missile turn rate. In addition, we specify that the turn rate U_m is the output of a first-order lag

$$\dot{U}_m=-p[U_m-U_c]$$

where U_c is the command acceleration. Define the state vector as $x'=[\delta x, \delta y, \delta\beta, U_m]$, and the control $u=U_c$ then

$$\dot{x}=Ax+Bu$$

‡This special form is convenient but not essential to perform the reduction in dimensionality. Any factorization of the form $M=C^TC$ where C is kxn can be used in the reduction.

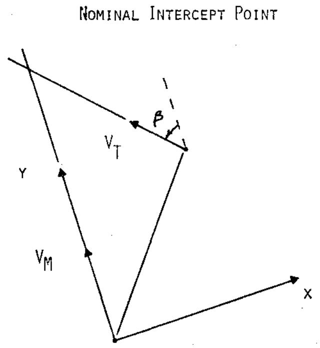

Fig. 2 Coordinate system for missile pursuit problem.

with

$$A=\begin{bmatrix} 0 & 0 & -V_T\cos\beta_0 & y_0(t) \\ 0 & 0 & -V_T\sin\beta_0 & -x_0(t) \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -p \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ p \end{bmatrix}$$

The control problem is to find the control U_c to minimize the miss distance $(\delta x_f^2+\delta y_f^2)$ subject to a quadratic penalty on the control. Thus

$$J=\frac{1}{2}x_f'Mx_f+\frac{1}{2}\int_0^{t_f}u_c^2dt$$

where

$$M=\begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case with $k=2$, $n=4$, use the reduced order transformation equations to obtain

$$\Lambda_R(\tau)=\begin{bmatrix} 1 & 0 & -V_T\cos\beta_0\tau & \frac{V_m}{p}\left[\tau+\frac{1}{p}(e^{-p\tau}-1)\right] \\ 0 & 1 & -V_T\sin\beta_0\tau & 0 \end{bmatrix}$$

where $\tau=t_f-t$

Substituting into Eq. (31), we find that

$$B_3(\tau)=\begin{bmatrix} V_m[\tau+\frac{1}{p}(e^{-p\tau}-1)] \\ 0 \end{bmatrix}$$

hence the y component of the miss is uncontrollable in this linearized model and the problem can be reduced to a first-order problem by ignoring the δy motion.

We thus have

$$\dot\eta=V_m\left[\tau+\frac{1}{p}(e^{-p\tau}-1)\right]U_c$$

where η is the projected miss in the x direction. The optimal control U_c^* can be calculated as

$$U_c^*=-V_m\left[\tau+\frac{1}{p}(e^{-p\tau}-1)\right]S(\tau)\eta(\tau)$$

where

$$S(\tau)=c_1\Big/\left(1+c_1\int_0^{\tau}B_3^2(\tau)d\tau\right)$$

$$=c_1\Big/\left(1+c_1V_m^2\left[\frac{\tau^3}{3}+\frac{1}{p^3}\left(\frac{7}{2}+p\tau-p^2\tau^2-2p\tau e^{-p\tau}-\frac{1}{2}e^{-2p\tau}\right)\right]\right)$$

VI. Conclusion

Use of a transformation approach for the terminal control problem yields insight into the structure of the optimal controller, and in many uses yields reduced computational effort. In many cases, use of the transformation method will allow analytical solution to problems otherwise intractable with the conventional Riccati approach.

References

[1]Bryson, A. E. and Ho, Y. C., *Applied Optimal Control,* Blaisdell Pub., Waltham, Mass., 1969.

[2]Seal, C. E. and Stubberud, A., "Final Value Control Systems," *Advances in Control Systems,* Vol. 8, edited by C. T. Leondes, Academic Press, N. Y., 1971, pp. 23-52.

[3]Slater, G. L. and Wells, W. R., "On the Reduction of Optimal Time Delay Systems to Ordinary Ones," *IEEE Transactions on Automatic Control,* Vol. AC-17, #1, Feb. 1972, pp. 154-155.

[4]Pontryagin, L. S., et al., *The Mathematical Theory of Optimal Processes,*" Interscience, N. Y., 1962.

[5]Slater, G. L., "Applications of Optimal Control to the Air-to-Air Tactical Missile Problem," Math Lab. Preprint Ser. #5, Applied Mathematics Research Lab., ARL, Wright-Patterson AFB, Ohio.